A basic two-state model for bosonic field theories with a cubic nonlinearity

2005 J. Phys. A: Math. Gen. 38 3505

(http://iopscience.iop.org/0305-4470/38/16/003)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 3505–3516 doi:10.1088/0305-4470/38/16/003

A basic two-state model for bosonic field theories with a cubic nonlinearity

Artur Ishkhanyan[1,3], **Juha Javanainen**[2] **and Hiroki Nakamura**[3]

[1] Engineering Center of Armenian NAS, Ashtarak-2, 378410, Armenia
[2] University of Connecticut, Storrs, CT 06269-3046, USA
[3] Institute for Molecular Science, Okazaki 444-8585, Japan

E-mail: artur@dolphin.am

Received 1 July 2004, in final form 18 January 2005
Published 6 April 2005
Online at stacks.iop.org/JPhysA/38/3505

Abstract

A basic nonlinear two-state model generic in classical and bosonic field theories with a cubic nonlinearity is considered. For the class of models with constant external field amplitude a general strategy for attacking the problem is developed based on the reduction of the initial system of equations for the semi-classical atom–molecule amplitudes to a nonlinear Volterra integral equation for the molecular probability. A uniformly convergent series solution to the problem is constructed for the weak interaction limit. The Landau–Zener model is considered as a specific example. The first approximation term is derived and an asymptotic expression for the nonlinear transition probability is established in the weak interaction regime.

PACS numbers: 32.80.Bx, 33.80.Be, 34.50.Rk, 03.75.Nt

1. Introduction

In the studies of photoassociation of a Bose–Einstein condensate [1], the following system of semiclassical *nonlinear* equations describing atomic and molecular condensates as classical fields has been derived [2, 3]:

$$\mathrm{i}\frac{\mathrm{d}a_1}{\mathrm{d}t} = U(t)\,\mathrm{e}^{-\mathrm{i}\delta(t)}a_2\overline{a}_1, \qquad \mathrm{i}\frac{\mathrm{d}a_2}{\mathrm{d}t} = \frac{U(t)}{2}\,\mathrm{e}^{\mathrm{i}\delta(t)}a_1a_1. \tag{1}$$

Here a_1 and a_2 are the atomic and molecular states' amplitudes, respectively, $\overline{a}_1$ is the complex conjugate to a_1, $U = U(t)$ is the Rabi frequency of the photoassociating laser field and $\delta = \delta(t)$ is the corresponding frequency detuning modulation function. The same equations will come up in an attempt to control the scattering length of an atomic Bose–Einstein condensate by means of a Feshbach resonance [4] (the Rabi frequency is then proportional to the square root of the magnetic-field width of the resonance, and the detuning modulation is proportional to the external magnetic field), in second-harmonic generation in nonlinear optics [5] (here U is proportional to the second-order susceptibility of a lossless quadratic medium and δ is the spatial phase difference between fundamental and second-harmonic waves), and generally in field theories where the system Hamiltonian is of the generic form $a_2^{\dagger}a_1a_1$. Because of the nonlinearity, a physical system that is governed by these equations demonstrates very complicated behaviour (this has been well revealed already in nonlinear optics studies—see, e.g., [5]). Compared with the linear quantum two-state problem [6], system (1) presents a much more complicated, qualitatively different mathematical problem (even in the simplest case of the Rabi problem [6], when both the Rabi frequency and detuning modulation function are constants, one faces essential complications [5], see also [7]) and at the present time, no general analytic approaches for treating such systems are known. In this paper, we propose an approach that is applicable to all the models with constant field amplitude. We demonstrate the effectiveness of the developed approach by considering the Landau–Zener model [8].

Elimination of a_1 leads to a second-order nonlinear ordinary differential equation for a_2:

$$\frac{\mathrm{d}^2a_2}{\mathrm{d}t^2} + \left(-\mathrm{i}\delta_t - \frac{U_t}{U}\right)\frac{\mathrm{d}a_2}{\mathrm{d}t} + U^2(1-2|a_2|^2)a_2 = 0, \tag{2}$$

where U is assumed to be real.

System (1) possesses a first integral

$$|a_1|^2 + 2|a_2|^2 = I_N = \text{const}. \tag{3}$$

We are interested in the solutions of (1) that belong to the manifold $I_N = 1$ and are defined by the initial conditions $|a_1(-\infty)|^2 = 1$, $|a_2(-\infty)|^2 = 0$. This normalization is incorporated in equation (2). The corresponding linear system has the form

$$\mathrm{i}\frac{\mathrm{d}a_1}{\mathrm{d}t} = U(t)\,\mathrm{e}^{-\mathrm{i}\delta(t)}a_2, \qquad \mathrm{i}\frac{\mathrm{d}a_2}{\mathrm{d}t} = U(t)\,\mathrm{e}^{\mathrm{i}\delta(t)}a_1, \tag{4}$$

the associated second-order linear ordinary differential equation for a_2 being (compare with equation (2))

$$\frac{\mathrm{d}^2a_2}{\mathrm{d}t^2} + \left(-\mathrm{i}\delta_t - \frac{U_t}{U}\right)\frac{\mathrm{d}a_2}{\mathrm{d}t} + U^2a_2 = 0. \tag{5}$$

The first integral of system (4) is $|a_1|^2 + |a_2|^2 = I_L$. Here of interest are the solutions that belong to the manifold $I_L = 1/4$ and satisfy initial conditions $|a_1(-\infty)|^2 = 1/4$, $|a_2(-\infty)|^2 = 0$. The reason for this choice is that then the solutions of nonlinear and linear problems asymptotically coincide at $t \to -\infty$.

We start with an important general observation that the solvable cases of system (1), like the linear case, fortunately, form a certain class [10], as can be easily verified by direct inspection. Namely, if the functions $a_{1,2}^*(z)$ are a solution of system (1) for some $U^*(z)$ and $\delta^*(z)$, the functions $a_{1,2}(t) = a_{1,2}^*(z(t))$ are then a solution to (1) for $U(t)$ and $\delta(t)$ given by $U(t) = U^*(z)\frac{\mathrm{d}z}{\mathrm{d}t}$, $\delta_t(t) = \delta_z^*(z)\frac{\mathrm{d}z}{\mathrm{d}t}$. This property allows one to write down the solution for any member of a class in terms of the solution for a certain basic representative of the class. Hence, due to this kind of property the number of different models to be considered is reduced to a handful of basic models. The Landau–Zener model subject to the treatment below is one of such basic models.

Further, in order to examine the role of the nonlinearity, we present a heuristic comparison of linear and nonlinear problems using the Landau–Zener model, $U = U_0 = \text{const}$, $\delta = \delta_0 t^2$, as a specific example of a model with constant field amplitude.

Linear solution. The Landau–Zener solution of system (4), often written using parabolic cylinder functions [6, 8], can more conveniently be expressed in terms of the Kummer confluent

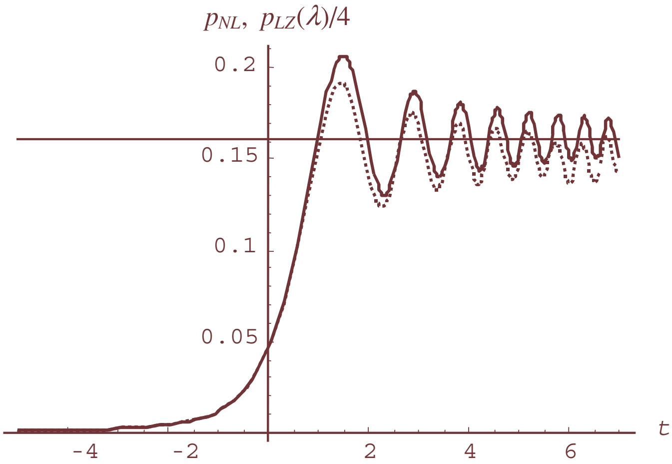

Figure 1. The behaviour of the solutions to the nonlinear (solid line) and linear (increased four times) Landau–Zener problems in the weak interaction regime, $\lambda \ll 1$.

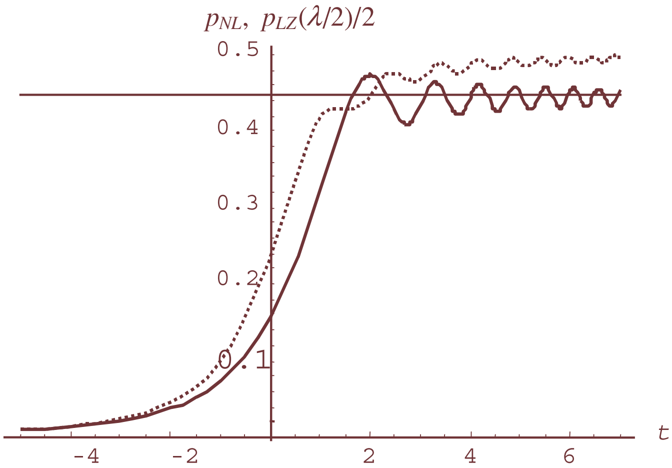

Figure 2. The behaviour of the solutions to the nonlinear (solid line) and linear (increased twice) Landau–Zener problems when the nonlinearity is strongly expressed, $\lambda \gg 1$.

hypergeometric function ${}_1F_1$ [9] as follows:

$$a_{2\mathrm{LZ}}(t) = C_1F_1 + C_2F_2, \qquad F_1 = {}_1F_1(\mathrm{i}\lambda/4, 1/2, \mathrm{i}\delta_0 t^2), \qquad F_2 = t\,{}_1F_1(1/2 + \mathrm{i}\lambda/4, 3/2, \mathrm{i}\delta_0 t^2), \tag{6}$$

$$C_1 = \sqrt{\lambda\,\mathrm{e}^{-\pi\lambda/4}\cosh(\pi\lambda/4)}\,\frac{\mathrm{i}}{2}\,\frac{\Gamma(1/2 - \mathrm{i}\lambda/4)}{\Gamma(1 - \mathrm{i}\lambda/4)}, \qquad C_2 = \sqrt{\lambda\,\mathrm{e}^{-\pi\lambda/4}\cosh(\pi\lambda/4)}\sqrt{\mathrm{i}\delta_0}, \tag{7}$$

where Γ is the gamma function and $\lambda = U_0^2\big/\delta_0$ is the Landau–Zener parameter. The time evolution of the probability for the second state, $p(t) = |a_2(t)|^2$, is shown in figures 1 and 2. At $t = +\infty$ we have the familiar Landau–Zener result

$$P_{\mathrm{LZ}} = 1 - \mathrm{e}^{-\pi\lambda}. \tag{8}$$

Nonlinear case. Initial intuitive insight into the problem is gained by examining equation (2). As is seen, the nonlinearity is of a local character, i.e. it is determined by the current value of the transition probability $p(t)$. Hence, one may expect that if $p(t)$ remains small enough (note that because of the normalization constraint (4), p is always not more than $1/2$) then the role of the nonlinearity is rather restricted. In this case, neglecting the nonlinear term in equation (2), we get a linear equation that is simply satisfied by a scaled linear solution: $a_2 = a_{2L}/2$. Now, the solution to the linear Landau–Zener problem (6) suggests that the nonlinear term remains small for all the time if the final population probability of the second state calculated from the linear solution is small. This is the case when the Landau–Zener parameter is much less than unity. We thus conclude that in the limit of small λ, the solution of the nonlinear problem is effectively the same as the scaled solution to the linear problem.

Suppose next that the nonlinear term becomes large, of the order of 1. The speculations above suggest that this is to be the case when $\lambda \gg 1$. Then dividing equation (2) by λ, we see that the small parameter $1/\lambda$ now stands at the highest-order derivative. Hence, this case is, generally speaking, of a singular nature and one may expect that the perturbation by the nonlinear term will cause essential deflections from the linear behaviour. These general speculations are further supported by numerical simulations. Qualitative comparison of the nonlinear and linear solutions is shown in figures 1 and 2. As is seen, the behaviours of the nonlinear and linear solutions are rather similar in the weak interaction regime: the differences seem to be only quantitative. However, in the strong interaction case we face essentially different asymptotes. Furthermore, the numerical simulations together with some analytic developments reveal the following facts [11]:

(i) in the weak interaction regime when the nonlinearity is less pronounced, $\lambda \ll 1$, the final transition probability can be presented as

$$p(+\infty) \approx \frac{P_{\mathrm{LZ}}(\lambda)}{4}\left(1 + \frac{\lambda}{\pi} P_{\mathrm{LZ}}(\lambda)\right), \tag{9}$$

(ii) in the strongly nonlinear limit, $\lambda \gg 1$, the final probability is approximately given as

$$p(+\infty) \approx \frac{P_{\mathrm{LZ}}(\lambda/2)}{2}\left(1 - \frac{4}{3\pi\lambda} P_{\mathrm{LZ}}(\lambda/2)\right). \tag{10}$$

Note that in the case of strong interactions, the final transition probability is interestingly related to the linear Landau–Zener formula with the parameter λ replaced by $\lambda/2$. It should be noted, however, that this is the case only for the final probability at $t \to +\infty$, not for the intermediate time evolution of the system, see also figure 2.

Now, the examination of the structure of formulae (9) and (10) might suggest that the nonlinearity presents a small perturbation, so that one may try to construct an approximate solution to equation (2) as a power series expansion in terms of λ if the Landau–Zener parameter is small and in terms of $1/\lambda$ in the opposite limit. However, these attempts to treat the nonlinearity by direct perturbation methods using the corresponding linear Landau–Zener solutions as zeroth-order approximations fail in both cases. It appears that the resulting approximation terms diverge even in the limit of weak interaction. This is probably not surprising at all, if we recall the experience accumulated in the study of nonlinear systems (see, e.g., [12] and numerous references therein).

Indeed, consider, e.g., the variation of parameters method [12]. Accordingly, the first-order approximation is written in terms of an arbitrary set of linearly independent fundamental solutions u_1 and u_2 to unperturbed linear homogeneous equation as follows:

$$a_2 = \frac{a_{2\mathrm{LZ}}(t)}{2} + \lambda\left(u_2(t)\int_{-\infty}^{t} \frac{u_1(x)}{W(x)} G(x)\,\mathrm{d}x - u_1(t)\int_{-\infty}^{t} \frac{u_2(x)}{W(x)} G(x)\,\mathrm{d}x\right), \tag{11}$$

where $W(x)$ is the Wronskian defined as $W = u_1u_2' - u_2u_1'$ and $G(x)$ is the inhomogeneous term,

$$G(x) = \tfrac{1}{4}|a_{2\mathrm{LZ}}|^2 a_{2\mathrm{LZ}}, \tag{12}$$

$a_{2\mathrm{LZ}}$ being the above solution to the linear Landau–Zener problem. Now consider the pair of independent solutions $u_{1,2} = C_1F_1 \pm C_2F_2$, with $F_{1,2}$ and $C_{1,2}$ defined by equations (6) and (7), so that $u_1 = a_{2\mathrm{LZ}}(t)$ and $u_2 = a_{2\mathrm{LZ}}(-t)$. At $t \to +\infty$ these functions have the asymptotes

$$u_1|_{t\to+\infty} = A_1 t^{-\mathrm{i}\lambda/2} + B_1 \frac{\mathrm{e}^{\mathrm{i}\delta_0 t^2} t^{+\mathrm{i}\lambda/2}}{t} + O\left(\frac{1}{t^2}\right), \tag{13}$$

$$u_2|_{t\to+\infty} = B_2 \frac{\mathrm{e}^{\mathrm{i}\delta_0 t^2} t^{+\mathrm{i}\lambda/2}}{t} + O\left(\frac{1}{t^2}\right), \tag{14}$$

A_1 and $B_{1,2}$ being (nonzero) constants. Since the Wronskian is $W = W_0\,\mathrm{e}^{\mathrm{i}\delta_0 t^2}$, where W_0 is a constant, we have the following asymptotes for the integrands involved in equation (11):

$$\frac{u_1(t)}{W(t)}G(t)|_{t\to+\infty} = \frac{1}{4W_0}\left[\left(|A_1|^2 A_1^2 t^{-\mathrm{i}\lambda}\,\mathrm{e}^{-\mathrm{i}\delta_0 t^2} + A_1^3 B_1^*\,\mathrm{e}^{-2\mathrm{i}\delta_0 t^2}\frac{t^{-2\mathrm{i}\lambda}}{t}\right) + \frac{3|A_1|^2 A_1 B_1}{t} + O\left(\frac{1}{t^2}\right)\right], \tag{15}$$

$$\frac{u_2(t)}{W(t)}G(t)|_{t\to+\infty} = \frac{1}{4W_0}\frac{|A_1|^2 A_1 B_2}{t} + O\left(\frac{1}{t^2}\right). \tag{16}$$

Hence, both integrals involved in equation (11) diverge logarithmically. As is immediately seen, while the second term of equation (11) remains finite all the time due to the asymptote of u_2, the last term is divergent.

It is thus now suspected that this is purely related to phase effects which do not show up when dealing with probabilities. One of the possibilities of checking this is to turn to an equation involving only probability p. As a result we get differential equations of the third order. For the particular case of the Landau–Zener model we are concerned with, the equation for the molecular state reads

$$p''' - \frac{p''}{t} + 4[t^2 + \lambda(1 - 3p)]p' + \frac{\lambda}{2t}(1 - 8p + 12p^2) = 0. \tag{17}$$

However, it can be verified that here we again face the same difficulties with divergences. This means that we have to employ some other non-trivial perturbation techniques such as the Krilov–Bogoliubov–Mitropolski averaging or the multiple-scale method which have proven to be highly successful in treating numerous problems in many branches of physics and mathematics [12]. This is actually the approach we used in [11]; but we should mention that it is too complicated, since higher transcendental functions are involved because equation (17) is of the third order.

Nevertheless, expressions (9) and (10) still suggest that, at least in the limit of weak interaction, $\lambda \ll 1$, when the nonlinear term presents a weak regular perturbation [12], some simpler perturbative approaches should be possible. In the next section we demonstrate that this is, indeed, the case. We derive a nonlinear Volterra integral equation [13], equivalent to equation (17), that allows one to avoid the divergence and to eventually construct uniformly convergent series solution for the case of small λ. Notably, this reduction is possible for all the analogous models with fixed field amplitude. Hence, this approach can be a general strategy for attacking analogous nonlinear two-state problems. It is demonstrated that a simple formula for the first correction term is derived and the final transition probability to the molecular state for the Landau–Zener problem can be calculated.

2. Nonlinear Volterra integral equation

Consider the case of constant field amplitude, $U = U_0 = \text{const}$, and as a first step, turn to the modulus and argument of the variables involved: $a_1(t) = r_1(t)\,\mathrm{e}^{\mathrm{i}\theta_1(t)}$ and $a_2(t) = r_2(t)\,\mathrm{e}^{\mathrm{i}\theta_2(t)}$. Taking the square of the modulus of the second equation of system (1), we easily get the following equation:

$$\left(\frac{\mathrm{d}r_2}{\mathrm{d}t}\right)^2 + r_2^2\left(\frac{\mathrm{d}\theta_2}{\mathrm{d}t}\right)^2 = \frac{U_0^2}{4}\left(1-2r_2^2\right)^2. \tag{18}$$

Denoting $p = r_2^2$ and $q = 2r_2^2\,\mathrm{d}\theta_2/\mathrm{d}t$ from equation (2) we can derive the following system:

$$\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)^2 + q^2 = U_0^2(1-2p)^2 p, \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = \delta_t\frac{\mathrm{d}p}{\mathrm{d}t}. \tag{19}$$

Compare this with the corresponding equations for the linear case:

$$\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)^2 + q^2 = U_0^2(1-4p)p, \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = \delta_t\frac{\mathrm{d}p}{\mathrm{d}t}. \tag{20}$$

(Note that equations (19) and (20) are rather convenient for generation of a number of exactly solvable models for both linear and nonlinear problems. Some examples are presented in our previous paper [14].)

After differentiation and some straightforward transformations, the first equation of (19) can be changed to

$$\frac{\mathrm{d}^2q}{\mathrm{d}t^2} - \frac{\delta_{tt}}{\delta_t}\frac{\mathrm{d}q}{\mathrm{d}t} + \delta_t^2 q = \frac{U_0^2\delta_t}{2}(1-8p+12p^2). \tag{21}$$

The general solution of the homogeneous equation (i.e. with the right-hand side equal to zero) is easily found:

$$q = C_1\cos(\delta(t)) + C_2\sin(\delta(t)). \tag{22}$$

The Wronskian $q_1q_{2t} - q_2q_{1t}$ of the fundamental solutions $q_1 = \cos(\delta)$ and $q_2 = \sin(\delta)$ is simply δ_t so that the general solution of the full inhomogeneous equation is written as

$$\begin{aligned} q = {} & \cos(\delta(t))\left(C_1 - \int_{-\infty}^{t}\sin(\delta(x))\frac{U_0^2}{2}(1-8p+12p^2)\,\mathrm{d}x\right) \\ & + \sin(\delta(t))\left(C_2 + \int_{-\infty}^{t}\cos(\delta(x))\frac{U_0^2}{2}(1-8p+12p^2)\,\mathrm{d}x\right), \end{aligned} \tag{23}$$

where the argument of the integrand is changed to x. In the force of the relation $\mathrm{d}q/\mathrm{d}t = \delta_t\,\mathrm{d}p/\mathrm{d}t$, the differentiation of this equation leads to an integro-differential equation for p, namely

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}t} = {} & -C_1\sin(\delta) + C_2\cos(\delta) + \cos(\delta)\int_{-\infty}^{t}\cos(\delta)\frac{U_0^2}{2}(1-8p+12p^2)\,\mathrm{d}x \\ & + \sin(\delta)\int_{-\infty}^{t}\sin(\delta)\frac{U_0^2}{2}(1-8p+12p^2)\,\mathrm{d}x. \end{aligned} \tag{24}$$

Since we are searching for the solution satisfying the initial condition $p(-\infty) = 0$, the constants C_1 and C_2 actually vanish.

Thus the equation for the transition probability is

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{U_0^2}{2}\left\{\cos(\delta)\int_{-\infty}^{t}\cos(\delta)(1-8p+12p^2)\,\mathrm{d}x + \sin(\delta)\int_{-\infty}^{t}\sin(\delta)(1-8p+12p^2)\,\mathrm{d}x\right\}. \tag{25}$$

Of course, the same procedure can be performed with the linear system to derive an integral equation where the terms proportional to p^2 are missing.

Now, integrating equation (25) we get

$$p(t) = \frac{U_0^2}{2}\int_{-\infty}^{t}\cos(\delta(x))\left(\int_{-\infty}^{x}\cos(\delta(y))(1-8p(y)+12p^2(y))\,\mathrm{d}y\right)\mathrm{d}x \\ + \frac{U_0^2}{2}\int_{-\infty}^{t}\sin(\delta(x))\left(\int_{-\infty}^{x}\sin(\delta(y))(1-8p(y)+12p^2(y))\,\mathrm{d}y\right)\mathrm{d}x. \quad (26)$$

Finally, the integration by parts leads to a nonlinear Volterra integral equation [13],

$$p(t) = \frac{\lambda}{2}\int_{-\infty}^{t}K(t,x)(1-8p(x)+12p^2(x))\,\mathrm{d}x, \quad (27)$$

where λ is a 'Landau–Zener parameter' defined as $\lambda = U_0^2\tau_0^2$ (here τ_0 is the time scale defined by a scaling transformation $t \to \tau_0 t$; in the case of the Landau–Zener model $\tau_0 = 1/\sqrt{\delta_0}$), and the kernel, $K(t,x)$, is given by

$$K(t,x) = (C_\delta(t) - C_\delta(x))\cos(\delta(x)) + (S_\delta(t) - S_\delta(x))\sin(\delta(x)), \quad (28)$$

with functions C_δ and S_δ defined as

$$C_\delta(t) = \int_{-\infty}^{t}\cos(\delta(x))\,\mathrm{d}x, \qquad S_\delta(t) = \int_{-\infty}^{t}\sin(\delta(x))\,\mathrm{d}x. \quad (29)$$

It can be readily shown by differentiation that the derived integral equation, (27), is equivalent to the third-order differential equation (17).

Working out the first term of the integrand in (27) gives the following Volterra equation of the second kind

$$p(t) = \frac{\lambda}{4}f(t) - 4\lambda\int_{-\infty}^{t}K(t,x)\left(p(x) - \frac{3}{2}p^2(x)\right)\mathrm{d}x, \quad (30)$$

where $f(t)$ involved in the *forcing function* [13] is given as

$$f(t) = C_\delta^2(t) + S_\delta^2(t). \quad (31)$$

Now, if the forcing function, i.e., effectively, $f(t)$, and the kernel, $K(t,x)$, are bounded, one may apply the Picard's successive approximations,

$$p_0 = \frac{\lambda}{4}f(t), \qquad p_n = \frac{\lambda}{4}f(t) - 4\lambda\int_{-\infty}^{t}K(t,x)\left(p_{n-1} - \frac{3}{2}p_{n-1}^2\right)\mathrm{d}x, \quad n \geqslant 1, \quad (32)$$

to construct a sequence of functions $p_n(t)$, which, according to the general theory (see, e.g., [13]), converges uniformly everywhere to a limit function $p(t)$ that is the unique solution to equation (30). Since the conditions used are quite general, the Volterra integral equation (27) or (30) may provide a systematic way to attack the nonlinear system (1) in the case of small enough Landau–Zener parameter λ.

However, the convergence of Picard's series (32) is very slow. To demonstrate this, note that by rearrangement of the terms the Picard's solution can be presented as a power series expansion in λ. This expansion accounts for the orders of the involved terms explicitly. Substituting $p = p_0 + \lambda p_1 + \lambda^2 p_2 + \cdots$ into equation (27) and equating coefficients at the same powers of λ we get $p_0 = 0$ and, successively,

$$\lambda : p_1 = \frac{1}{2}\int_{-\infty}^{t}K(t,x)\,\mathrm{d}x, \qquad \lambda^2 : p_2 = \frac{1}{2}\int_{-\infty}^{t}K(t,x)(-8p_1)\,\mathrm{d}x, \quad (33)$$

$$\lambda^3 : p_3 = \frac{1}{2}\int_{-\infty}^{t}K(t,x)\big(-8p_2 + 12p_1^2\big)\,\mathrm{d}x, \quad \ldots. \quad (34)$$

(Of course, λp_1 is the forcing function of the Volterra equation (30): $p_1 = \left[C_\delta^2(t) + S_\delta^2(t)\right]/4 = f(t)/4$.) Now, since all the p_i tend to finite, in general nonzero values at $t \to +\infty$, it is understood that any finite sum, being a polynomial in λ, is not restricted at $\lambda \to \infty$. Furthermore, it can even take negative values. For instance, for the Landau–Zener model $p_1 = f(t)/4 \to \pi/4$ at $t \to +\infty$ so that λp_1 starting from $\lambda \approx 0.65$ already exceeds the maximum $1/2$ allowed by the normalization (3), and the next approximation, $\lambda p_1 + \lambda^2 p_2$, becomes less than zero when $\lambda > 0.65$.

Thus another approach is preferred. Note first that p_0, p_1 and p_2 satisfy the same equations as the corresponding terms of the expansion in the linear case. Hence, we are lead to apply to the initial integral equation (27) the substitution $p = p_L + u$, p_L being the scaled linear solution (i.e., the linear solution with normalization $I_L = 1/4$). Then we have

$$p_L + u = \frac{\lambda}{2}\int_{-\infty}^{t} K(t,x)[1 - 8(p_L + u) + 12(p_L + u)^2(x)]\,\mathrm{d}x. \tag{35}$$

Cancelling the terms belonging to the linear problem leads to a new Volterra integral equation of Hammerstein type [13],

$$u = 6\lambda \int_{-\infty}^{t} K(t,x) p_L^2\,\mathrm{d}x - 4\lambda \int_{-\infty}^{t} K(t,x)\left[(1 - 3p_L)u - \frac{3}{2}u^2\right]\mathrm{d}x, \tag{36}$$

with changed forcing function that is of the order of λ^3. It is then understood that this forcing function should lead to much faster converging approximations. Indeed, try now an expansion of the form

$$u = u_0 + \lambda u_1 + \lambda^2 u_2 + \lambda^3 u_3 + \cdots. \tag{37}$$

Since $p_L \sim \lambda$ at small λ, we conclude that $u_0 = u_1 = u_2 = 0$. For the next term, however, we get an important result:

$$\lambda^3 u_3 = 6\lambda \int_{-\infty}^{t} K(t,x) p_L^2\,\mathrm{d}x. \tag{38}$$

Noting that $u = \lambda^3 u_3 + O(\lambda^4)$, we finally arrive at a principal result:

$$u \approx 6\lambda \int_{-\infty}^{t} K(t,x) p_L^2\,\mathrm{d}x. \tag{39}$$

This is the desired form of the first correction term. Thus, finally, we obtain that in the first approximation the solution to the nonlinear problem in the weak interaction regime is written as

$$p(t) = p_L(t) + 6\lambda \int_{-\infty}^{t} K(t,x)[p_L(x)]^2\,\mathrm{d}x. \tag{40}$$

This is numerically proven to be a good approximation. For the Landau–Zener model, up to $\lambda < 0.5$ the comparison with the numerical solution to system (1) gives practically indistinguishable graphs. And it also works well as a first approximation even up to $\lambda \lesssim 1$.

3. Landau–Zener model: final transition probability

Consider the Landau–Zener model:

$$\delta = \delta_0 t^2. \tag{41}$$

The natural time scale here is $\tau_0 = 1/\sqrt{\delta_0}$ so that λ becomes the conventional Landau–Zener parameter: $\lambda = U_0^2/\delta_0$. Functions $C_\delta(t)$ and $S_\delta(x)$ get the form

$$C_\delta(t) = \sqrt{\frac{\pi}{2\delta_0}}\left[\frac{1}{2} + C\left(\sqrt{\frac{2\delta_0}{\pi}}t\right)\right], \qquad S_\delta(t) = \sqrt{\frac{\pi}{2\delta_0}}\left[\frac{1}{2} + S\left(\sqrt{\frac{2\delta_0}{\pi}}t\right)\right], \tag{42}$$

with C and S being the Fresnel functions [9] defined as

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}\xi^2\right) \mathrm{d}\xi, \qquad S(x) = \int_0^x \sin\left(\frac{\pi}{2}\xi^2\right) \mathrm{d}\xi. \tag{43}$$

Interestingly, the forcing function of the Volterra equation (30) then becomes a prominent function from the theory of light diffraction:

$$f(t) = \frac{\pi}{2\delta_0}\left\{\left[\frac{1}{2} + C\left(\sqrt{\frac{2\delta_0}{\pi}}t\right)\right]^2 + \left[\frac{1}{2} + S\left(\sqrt{\frac{2\delta_0}{\pi}}t\right)\right]^2\right\}. \tag{44}$$

As is well known, this function defines the light intensity behind a semi-infinite opaque wall, t playing then the role of the lateral distance from the end of the wall [15].

Let us now calculate the final transition probability in the limit $t \to \infty$:

$$p(+\infty) = \frac{p_{\mathrm{LZ}}(+\infty)}{4} + u(+\infty), \qquad u(+\infty) = 6\lambda \int_{-\infty}^{+\infty}\left[\left(\sqrt{\frac{\pi}{2}} - C_\delta(x)\right)\cos\delta(x) + \left(\sqrt{\frac{\pi}{2}} - S_\delta(x)\right)\sin\delta(x)\right][p_L(x)]^2\,\mathrm{d}x. \tag{45}$$

Since the correction term u is of the order of λ^3, we first can get an initial estimate by means of replacing p_L by $\lambda p_1 = \lambda f(t)/4$. In this case we obtain that at infinity

$$\begin{aligned} p(+\infty) &= p_L(+\infty) + \lambda^3\left(-\frac{\pi^3}{16} + \frac{3}{8}\sqrt{\frac{\pi}{2}}I_G\right), \\ I_G &= \int_{-\infty}^{+\infty}(\cos(\delta_0 x^2) + \sin(\delta_0 x^2))\left(C_\delta^2(x) + S_\delta^2(x)\right)^2 \mathrm{d}x. \end{aligned} \tag{46}$$

Numerical integration results in $I_G = 5.8412$, so that we have

$$p(+\infty) = \frac{P_{\mathrm{LZ}}}{4} + u(+\infty) \approx \frac{P_{\mathrm{LZ}}}{4} + 0.80743\lambda^3. \tag{47}$$

As an initial approximation, this expression well confirms the result of [11], formula (9). Indeed, the latter leads to

$$p(+\infty) = \frac{P_{\mathrm{LZ}}}{4} + \frac{\pi}{4}\lambda^3 + O(\lambda^4). \tag{48}$$

So, already in this approximation the difference is quite small: $0.80743 - \pi/4 \approx 0.022$.

However, the result can be essentially improved. This can be done by noting that the linear solution p_L involved in the integral in (45) can be well approximated by a formula of the form:

$$p_L(t) \approx \frac{P_{\mathrm{LZ}}}{4} f_L(t), \tag{49}$$

where the function $f_L(t)$ does not depend on λ. The form of this function can be established from equation (17) as follows. Substituting $p = P_{\mathrm{final}} f_L(t)$, where P_{final} is the final transition probability at $t \to +\infty$, into equation (17) and then dividing it by P_{final} one obtains

$$f_L''' - \frac{f_L''}{t} + [4t^2 + 4\lambda(1 - 3P_{\mathrm{final}} f_L)]f_L' + \frac{\lambda}{2t}\left(\frac{1}{P_{\mathrm{final}}} - 8f_L + 12\frac{P_{\mathrm{final}}}{4}f_L^2\right) = 0. \tag{50}$$

Now we take the limit $\lambda \to 0$ keeping in mind that $\lambda/P_{\mathrm{final}} \approx \lambda/(P_{\mathrm{LZ}}/4) = 4\lambda/(1 - \mathrm{e}^{-\pi\lambda}) \to 4/\pi$ to derive an equation for the limit function $f_L(t)$:

$$f_L''' - \frac{f_L''}{t} + 4t^2 f_L' + \frac{2}{\pi t} = 0. \tag{51}$$

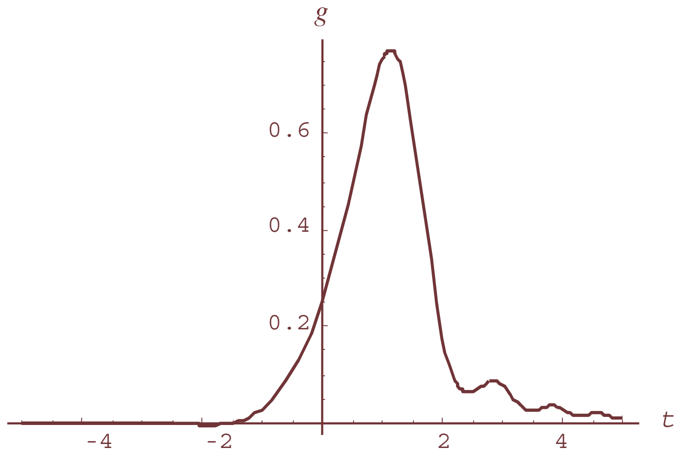

Figure 3. The behaviour of the integrand in formula (48).

The particular solution to this equation subject to the initial conditions considered here is

$$f_L(t) = -\frac{1}{4} + \frac{4(C_\delta + S_\delta)}{\pi^2} + \frac{t^2}{2\pi}[{}_2F_2(1,1;3/2,2;+\mathrm{i}t^2) + {}_2F_2(1,1;3/2,2;-\mathrm{i}t^2)]. \tag{52}$$

Inserting now equation (49) into equation (45) gives

$$p(+\infty) = \frac{P_{\mathrm{LZ}}}{4} + \lambda\left(\frac{P_{\mathrm{LZ}}}{4}\right)^2 I, \tag{53}$$

where

$$I = 6\int_{-\infty}^{+\infty}\left[\left(\sqrt{\frac{\pi}{2}} - C_\delta(x)\right)\cos\delta(x) + \left(\sqrt{\frac{\pi}{2}} - S_\delta(x)\right)\sin\delta(x)\right][f_L(x)]^2\,\mathrm{d}x. \tag{54}$$

The form of the integrand is shown in figure 3. As is immediately seen, the integrand effectively differs from zero only in a small interval near the origin. Though the analytic treatment here is straightforward, say, by using series expansions at the origin, it can be just calculated numerically since the integral is simply a number. The result is $I = 1.3317 \approx 4/3$. Direct simulations using the very linear Landau–Zener solution further improve the result to give $I = 1.3082$ to stand for I in (53), whereby we confirm the result (9) of [11] which reads $I = 4/\pi \approx 1.2732$. The derived formula (53) with the last number well agrees with the numerical solution: up to $\lambda \approx 0.4$ the relative error is less than 10^{-3}.

4. Summary

We have presented an analysis of a nonlinear version of the Landau–Zener problem that arises in different physical situations, e.g., in photoassociation of an atomic Bose–Einstein condensate, in controlling the scattering length of an atomic condensate by means of a Feshbach resonance, in second-harmonic generation, and generally in field theories with a cubic nonlinearity. We have shown that the governing equations can be reduced to the nonlinear Volterra equation that allows one to avoid the divergence problems which appear in the conventional perturbative approach. This equation allows one to construct uniformly convergent series solution for the case of small Landau–Zener parameter involved. We have derived the first correction term to the zeroth-order solution and have calculated the final transition probability for this case of weak interaction for the Landau–Zener model.

Notably, the reduction of the initial nonlinear two-state problem to the Volterra nonlinear integral equation is not restricted to the particular Landau–Zener problem treated here, but is the case for all the models with constant field amplitude. Furthermore, due to the above-mentioned class property of the solutions of the considered system, this reduction can be extended to the most of the possible models as well. Hence, the developed approach is a very general strategy for attacking analogous nonlinear two-state problems. Different useful models well known from standard linear theory are subject to treatment using this method. Of first interest could be the second Demkov–Kunike model [16] and the quadratic potential or double crossing model proven to be very helpful both in basic theory and in numerous important applications [6, 17]. The latter model in the linear version was solved by Zhu and Nakamura [18]. We would also like to mention possible three-mode and, generally, multi-mode extensions aggregating another block of problems attracting attention. Finally, an important further development could be the extension of the presented approach to the analogous models for systems including nonlinear terms associated with atom–atom, atom–molecule and molecule–molecule interactions. Such systems have recently gained much attention by various authors [4, 19, 20].

Acknowledgments

This work is supported partially by a Grant-in-Aid for Scientific Research on Specially Promoted Project no 15002011 from the Ministry of Education, Culture, Sports, Science, and Technology of Japan. AI thanks the Institute for Molecular Science, Okazaki National Research Institutes (Japan), where this research has been accomplished, for kind hospitality, Gennady Chernikov, Vazgen Ghazaryan and Ashot Manukyan for useful discussions, as well as acknowledges the support by the US Civilian Research and Development Foundation (grant NFSAT PH 100-02/12042) and Armenian National Science and Education Fund (grant ANSEF PS 10-2005).

References

[1] Anderson M H, Ensher J R, Matthews M R, Wieman C E and Cornell E A 1995 *Science* **269** 198
Davis K B, Mewes M-O, Andrews M R, van Druten N J, Durfee D S, Kurn D M and Ketterle W 1995 *Phys. Rev. Lett.* **75** 3969
Bradley C C, Sackett C A, Tollett J J and Hulet R G 1995 *Phys. Rev. Lett.* **75** 1687
Anglin J R and Ketterle W 2002 *Nature* **416** 211
[2] Javanainen J and Mackie M 1999 *Phys. Rev.* A **59** R3186
Kostrun M, Mackie M, Cote R and Javanainen J 2000 *Phys. Rev.* A **62** 063616
Mackie M and Javanainen J 1999 *Phys. Rev.* A **60** 3174
[3] Drummond P D, Kheruntsyan K V and He H 1998 *Phys. Rev. Lett.* **81** 3055
Heinzen D J, Wynar R, Drummond P D and Kheruntsyan K V 2000 *Phys. Rev. Lett.* **84** 5029
[4] Timmermans E, Tommasini P, Hussein M and Kernan A 1999 *Phys. Rep.* C **315** 199
van Abeelen F A and Verhaar B J 1999 *Phys. Rev. Lett.* **83** 1550
[5] Shen Y R 2002 *The Principles of Nonlinear Optics* (New York: Wiley)
Boyd R W 1992 *Nonlinear Optics* (Boston, MA: Academic)
Yariv A 1989 *Quantum Electronics* (New York: Wiley)
Butcher P N and Cotter D 1991 *The Elements of Nonlinear Optics* (Cambridge: Cambridge University Press)
Akhmanov S A and Khokhlov R V 1972 *Problems of Nonlinear Optics* (New York: Gordon and Breach)
[6] Shore B W 1990 *The Theory of Coherent Atomic Excitation* (New York: Wiley)
Nikitin E E and Umanski S Ya 1984 *Theory of Slow Atomic Collisions* (Berlin: Springer)
Vitanov N V and Garraway B M 1996 *Phys. Rev.* A **53** 4288
Vitanov N V and Garraway B M 1996 *Phys. Rev.* A **54** 5458(E)

[7] Ishkhanyan A, Chernikov G P and Nakamura H 2004 Rabi problem in photoassociation of atomic Bose–Einstein condensates *Phys. Rev.* A **70** 053611

[8] Landau L D 1932 *Phys. Z. Sowjetunion* **2** 46
Zener C 1932 *Proc. R. Soc.* A **137** 696

[9] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
Erdelyi A, Magnus W, Oberhettinger F and Tricomi F G 1953 *Higher Transcendental Functions* (New York: McGraw-Hill)
Slater L J 1966 *Generalized Hypergeometric Functions* (Cambridge: Cambridge University Press)

[10] Ishkhanyan A M 2000 *J. Phys. A: Math. Gen.* **33** 5539
Ishkhanyan A M 2000 *Opt. Commun.* **176** 155
Ishkhanyan A M 1997 *J. Phys. A: Math. Gen.* **30** 1203
Ishkhanyan A M and Suominen K-A 2001 *J. Phys. A: Math. Gen.* **34** 6301

[11] Ishkhanyan A, Mackie M, Carmichael A, Gould P L and Javanainen J 2004 Landau–Zener problem for trilinear systems *Phys. Rev.* A **69** 043612
Ishkhanyan A M and Chernikov G P 2004 Weak interaction limit in the theory of photoassociation of cold atoms *J. Contemp. Phys. (Armenian Academy of Science)* **39** 1

[12] Nayfeh A H 1985 *Perturbation Methods* (New York: Wiley)
Kevorkian J and Cole J D 1985 *Perturbation Methods in Applied Mathematics* (Heidelberg: Springer)
Kevorkian J and Cole J D 1996 *Multiple Scale and Singular Perturbation Methods* (New York: Springer)

[13] Tricomi F G 1985 *Integral Equations* (New York: Dover)
Brunner H and van der Houwen P J 1986 *The Numerical Solution of Volterra Equations* (Amsterdam: North Holland)
Burton T A 1983 *Volterra Integral and Differential Equations* (New York: Academic)
Baker C T H 1978 *The Numerical Treatment of Integral Equations* (Oxford: Clarendon)
Miller R K 1971 *Nonlinear Volterra Integral Equations* (New York: Benjamin)

[14] Ishkhanyan A, Mackie M, Gould Ph and Javanainen J 2003 *Interactions in Ultracold Gases: From Atoms to Molecules* ed M Weidemuller and C Zimmermann (Berlin: Wiley) p 470

[15] Sommerfeld A 1896 *Math. Ann.* **47** 317
Born M and Wolf E 1999 *Principles of Optics* (Cambridge: Cambridge University Press)

[16] Demkov Yu N and Kunike M 1969 *Vest. Leningr. Univ. Fiz. Khim* **16** 39
Suominen K-A and Garraway B M 1992 *Phys. Rev.* A **45** 374
Vitanov N V and Garraway B M 1996 *Phys. Rev.* A **53** 4288
Vitanov N V and Garraway B M 1996 *Phys. Rev.* A **54** 5458(E)

[17] Nakamura 2002 *Nonadiabatic Transition: Concepts, Basic Theories and Applications* (Singapore: World Scientific)
Teranishi Y and Nakamura H 1998 *Phys. Rev. Lett.* **81** 2032

[18] Zhu C, Teranishi Y and Nakamura H 2001 *Adv. Chem. Phys.* **117** 127
Teranishi Y and Nakamura H 1997 *J. Chem. Phys.* **107** 1904

[19] Yurovsky V A, Ben-Reuven A, Julienne P S and Williams C J 1999 *Phys. Rev.* A **60** R765
Yurovsky V A, Ben-Reuven A, Julienne P S and Williams C J 2000 *Phys. Rev.* A **62** 043605

[20] Weidemuller M and Zimmermann C (ed) 2003 *Interactions in Ultracold Gases: From Atoms to Molecules* (Berlin: Wiley)